Baily–Borel compactifications of period images and the *b*-semiampleness conjecture

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BGS
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$L = \det \pi_* \Omega_{Z/\mathcal{A}_g}$ extends to an ample bundle $L_{A_g^{\mathrm{SBB}}}$ (up to a power).

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(Satake '60, Baily–Borel '66) Generalized to arbitrary arithmetic locally symmetric varieties.

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$\left((X^\circ)^{\mathrm{an}} \xrightarrow{\psi} \mathcal{Y}\right) = \left(X^\circ \xrightarrow{f} Y\right)^{\mathrm{an}}$ and $L_{\mathcal{Y}} = (L_Y)^{\mathrm{an}}$ all algebraic, L_Y ample.

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- $B_Y := \bigoplus_k H^0_{mg}(Y, L_Y^k)$ *is finitely generated.*
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- Lots of previous work of Green–Griffiths–Laza–Robles and Green–Griffiths–Robles, including some special cases. Green–Griffiths–Robles establish key ingredient of our proof.

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This is a polarizable $\mathbb{Z}$-VHS whose Hodge bundle is the Griffiths bundle of V.

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Integrability. If period map of M_X is not generically immersive on some subvariety, then the period map of the $\mathbb{Q}$-closure is not generically immersive.

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Theorem (Green–Griffiths–Robles)

The Griffiths bundle has torsion combinatorial monodromy.

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Corollary

Moduli stacks of polarized klt CY pairs have canonical Baily–Borel compactifications, up to taking coarse space, reduction, normalization.

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Key: AND $(*) + (**) \Rightarrow$ connected comp. of fibers on $\widetilde{X_S}^{V_S^{\min}}$ are **compact**.

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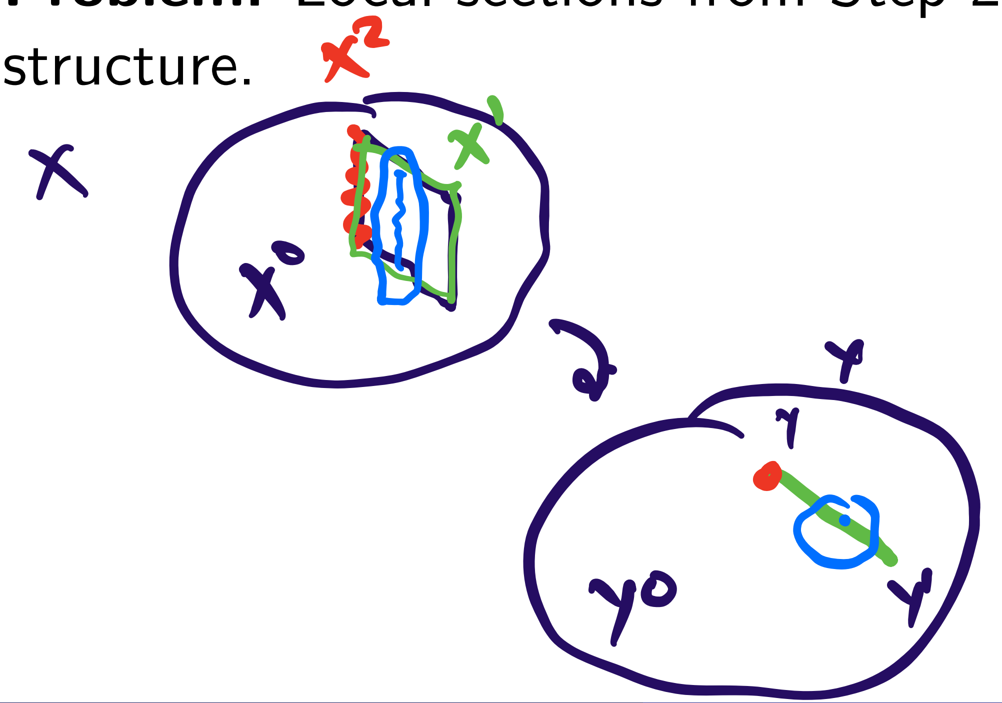

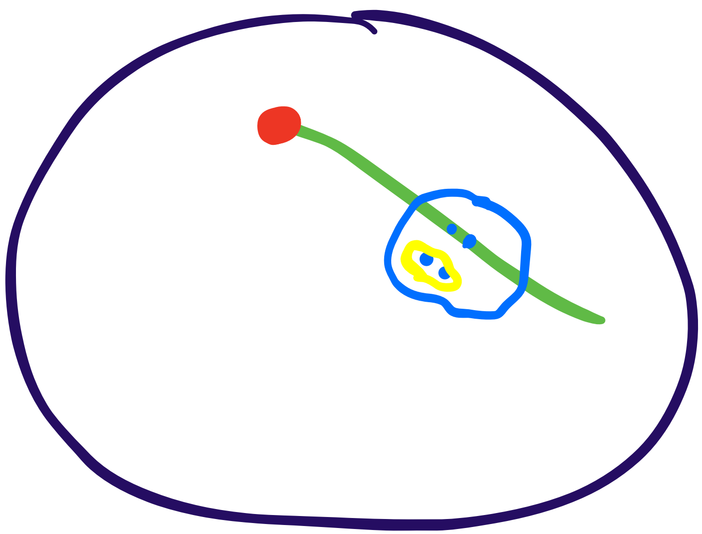

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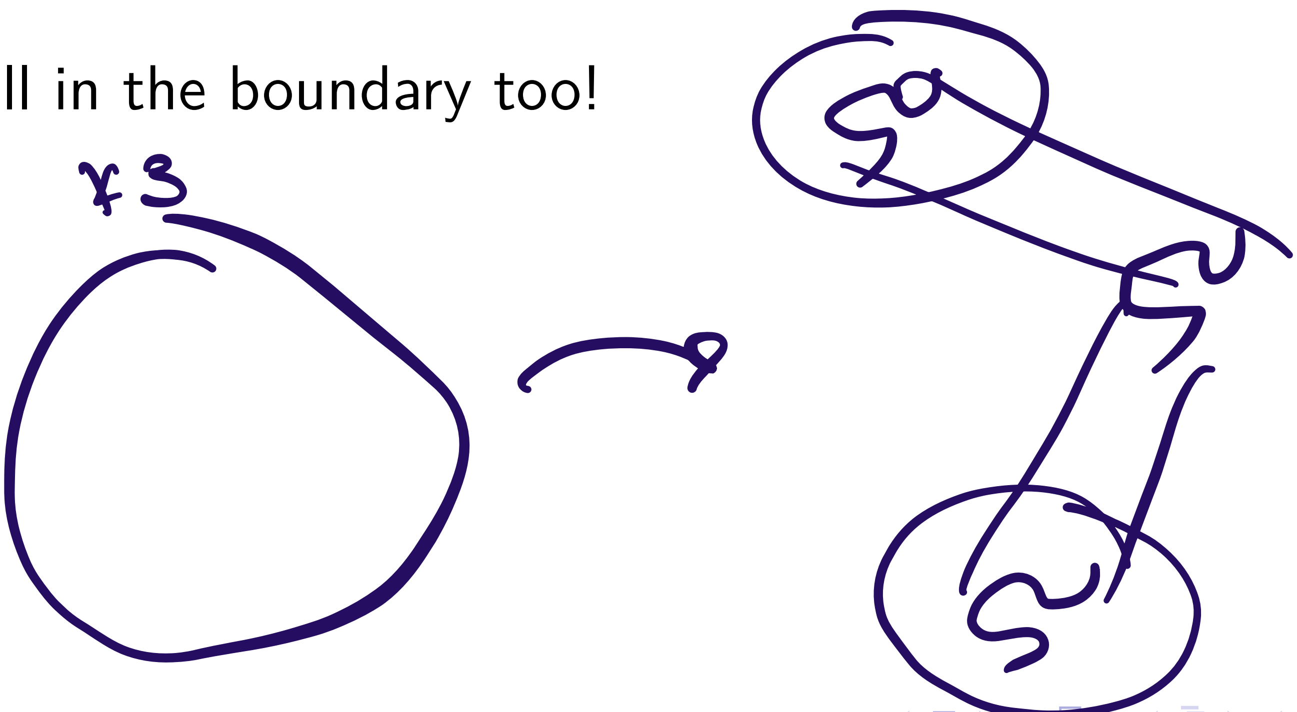
Key. Works well in the boundary too!



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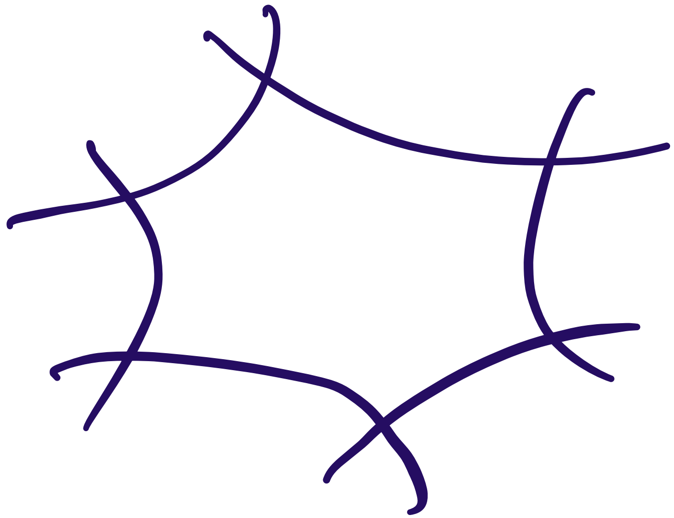
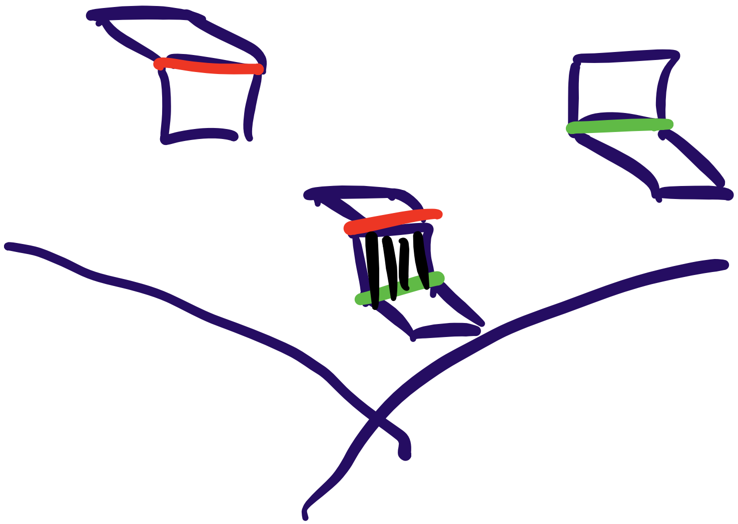
So if the Hodge bundle is trivial along a transcendental curve, the source must vary trivially, so V^{tr} is isotrivial.

Thm 3: torsion combinatorial monodromy





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$$\operatorname{img}\left(\mathrm{Bir}(S_1, S_2) \to \mathrm{Hom}(H^0(\omega_{S_1}), H^0(\omega_{S_2})\right) < \infty$$

Thanks!